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COMMENT

Comment on 'A simple proof of the Perron–Frobenius theorem for positive symmetric matrices'

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Abstract. We note that Ninio in his simple proof of the Perron–Frobenius theorem repeats in fact the Jentzsch proof.

Only recently an elementary proof of the Perron–Frobenius theorem for a positive symmetric matrix was given by Ninio (1976). This theorem guaranteed that the existence of the positive, non-degenerate eigenvalue would be greater than the absolute value of any other eigenvalues. This is useful in statistical mechanics when the transfer matrix method (see for example Domb 1960) is applied to obtain both the thermodynamic properties and the correlation functions. The partition function of a system can be expressed in the form (Domb 1960):

$$Z_N = \mathrm{Tr}(\mathbf{A}^N) = \sum_j \lambda_j^N$$

where λ_j are the eigenvalues of the transfer matrix $\mathbf{A}$. Due to the Perron–Frobenius theorem the free energy per particle is

$$f = -kT \ln \lambda_{\max}.$$

For classical systems the transfer matrix has to be replaced by an integral transfer operator with the kernel $A(\boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{x}$ stands generally for $(x_1, x_2, \ldots, x_n)$. The partition function for such systems is

$$Z_N = \int A_N(\boldsymbol{x}, \boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \sum_j \lambda_j^N,$$

where $A_N(\boldsymbol{x}, \boldsymbol{y})$ denotes the kernel of Nth iterate of the integral operator with the kernel $A(\boldsymbol{x}, \boldsymbol{y})$ and λ_j are the eigenvalues of the corresponding Fredholm equation

$$\int A(\boldsymbol{x}, \boldsymbol{y}) g(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \lambda g(\boldsymbol{x}).$$

In most applications to statistical mechanics the kernel $A(\boldsymbol{x}, \boldsymbol{y})$ is real, positive and symmetric. The analogue of the Perron–Frobenius theorem for this case is the Jentzsch theorem (Jentzsch 1912), which reads as follows.

If $A(\boldsymbol{x}, \boldsymbol{y})$ is a continuous, positive kernel and λ_{max} denotes its largest eigenvalue, then

(i) λ_{max} is positive,
(ii) λ_{max} is non-degenerate,
(iii) there exists a corresponding eigenfunction which is positive,
(iv) $\lambda_{max} > |\lambda|$, where λ is any other eigenvalue.

It is very easy to extend Ninio's proof for the classical systems. Then, this proof is, in fact, that given by Jentzsch (1912).

References

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